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On representations of the q-deformed Lorentz and Poincaré algebras

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Abstract. We construct explicitly all finite-dimensional representations of the quantum Lorentz group $SL_q(2, \mathbb{C})$. Based on this we prove that the q-deformed Lorentz algebra which was recently introduced can be considered as a Hopf algebra dual to $SL_q(2, \mathbb{C})$. The generators of the q-deformed Lorentz algebra act via a differential representation on a Hilbert space built of corepresentation spaces of the quantum Lorentz group. A chiral decomposition of the q-deformed Lorentz algebra is proposed. It is shown how spinor bases for the q-deformed Poincaré algebra can be constructed.

1. Introduction

The symmetries of flat spacetime play an essential role in physics. For example, laws of physics are required to be invariant under Lorentz transformations and the notion of an elementary particle can be understood within the representation theory of the Poincaré group. In recent years quantum groups have been investigated intensively. It has been shown that both Lorentz and Poincaré symmetry admit a generalization within the concept of quantum groups [1–5]. In this work we address some problems in the representation theory of the q-deformed Lorentz and Poincaré algebras.

The paper is outlined as follows. In section 2 we recall some facts about the representation theory of the quantum group $SU_q(2)$. It will turn out that the finite-dimensional corepresentations correspond to the space of undotted spinors of $SL_q(2, \mathbb{C})$. The space of dotted spinors or complex conjugate representations of the quantum Lorentz group (QLGr) are addressed in section 3. We will show in section 4 how these two representations can be combined in order to obtain all finite-dimensional representations of $SL_q(2, \mathbb{C})$. It is shown that the complex quantum plane [6, 7] is a specific corepresentation space of the QLG. In section 5 the concept of differential representations [8] is introduced which leads to Hilbert space representations of the compact part of the q-deformed Lorentz algebra. In the next section we prove that the q-Lorentz generators of [3] generate a Hopf algebra dual to the QLG of [2]. This analysis shows how a proper chiral decomposition of the q-Lorentz algebra can be obtained. Furthermore, we construct the Casimir elements of the q-deformed Lorentz algebra. The last section is devoted to the construction of irreducible massive spinor representations of the q-deformed Poincaré algebra.

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2. Representations of $SU_q(2)$

Let $\mathcal{G} := \mathbb{C}\langle a, b, c, d\rangle$ be the free associative $\mathbb{C}$-algebra in the variables a, b, c, d. $\mathcal{G}$ carries the structure of a bialgebra induced by the matrix

$$\left(u^i{}_j\right)_{i,j=1,2} := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1}$$

i.e. a comultiplication Δ and a counit ϵ are defined on the generators of $\mathcal{G}$ by

$$\Delta\left(u^i{}_j\right) = \sum_{k=1}^{2} u^i{}_k \otimes u^k{}_j \qquad \epsilon\left(u^i{}_j\right) = \delta^i{}_j. \tag{2}$$

I_q denotes the biideal in $\mathcal{G}$ which is generated by the well known relations [9]

$$\begin{aligned} &ab = qba \qquad bd = qdb \qquad bc = cb \\ &ac = qca \qquad cd = qdc \qquad ad = da + \lambda_q bc. \end{aligned} \tag{3}$$

The deformation parameter is chosen to be $q > 1$, and the abbreviation $\lambda_q = q - q^{-1}$ is sometimes used. The matrix u can be made unimodular by setting the quantum determinant $\det_q(u) := ad - qbc = 1$. We add this condition to the ideal I_q and define $\mathcal{G}_q := \mathcal{G}/I_q$. The bialgebra structure of $\mathcal{G}$ naturally extends to $\mathcal{G}_q$. It is possible to define an antipode S and an antimultiplicative involution $* : \mathcal{G}_q \longrightarrow \mathcal{G}_q$ by

$$\begin{aligned} &S\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix} \\ &\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* := \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} := \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix}. \end{aligned} \tag{4}$$

The choice of the involution is such that u becomes a unitary matrix, i.e. $u^* = S(u)$. As a consequence of these definition the algebra $\mathcal{A}_q := \left(\mathcal{G}_q, \Delta, \epsilon, S, *\right)$ is a $*$-Hopf algebra [10]. $\mathcal{A}_q$ is called quantum matrix group $SU_q(2)$. Since $\mathcal{A}_q$ is a $\mathbb{C}$-vector space it can be shown [11] that the set

$$\{a^i b^j c^k d^l \in \mathcal{A}_q : i, j, k, l \in \mathbb{N}_0,\ i = 0 \text{ or } l = 0\} \tag{5}$$

constitutes a $\mathbb{C}$-basis of $\mathcal{A}_q$. For later use we introduce [12] the mappings $P, Q : \mathcal{A}_q \longrightarrow \mathcal{A}_q$

$$P\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} a & c \\ b & d \end{pmatrix} \qquad Q\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} d & b \\ c & a \end{pmatrix}. \tag{6}$$

By definition, P is an algebra homomorphism whereas Q is an anti-algebra homomorphism. One easily proves the identities $P^2 = Q^2 = \mathrm{id}_{\mathcal{A}_q}$ and $PQ = QP$.

We now define $\mathbb{C}$-vector subspaces $V^L(l)$, $V^R(l)$, $\subset \mathcal{A}_q$, $l \in \frac{1}{2}\mathbb{N}_0$:

$$V^L(l) := \bigoplus_{i \in I_l} \mathbb{C}\xi_i^{(l)} \qquad \xi_i^{(l)} := \begin{bmatrix} 2l \\ l+i \end{bmatrix}_{q^{-2}}^{1/2} a^{l-i} c^{l+i} \tag{7}$$

$$V^R(l) := \bigoplus_{j \in I_l} \mathbb{C}\eta_j^{(l)} \qquad \eta_j^{(l)} := \begin{bmatrix} 2l \\ l+j \end{bmatrix}_{q^{-2}}^{1/2} a^{l-j} b^{l+j}. \tag{8}$$

Throughout this work we maintain the convention that the index set is $I_l := \{-l, -l+1, \ldots, l\}$, and the q-binomials are as usual [8] defined by

$$\begin{bmatrix} m \\ n \end{bmatrix}_{q^\alpha} := \frac{[m]_{q^\alpha}!}{[n]_{q^\alpha}![m-n]_{q^\alpha}!} \qquad \text{with } [n]_{q^\alpha} = \frac{1 - q^{\alpha n}}{1 - q^\alpha} \tag{9}$$

for some $m, n \in \mathbb{N}_0$.

These vector spaces can be identified with the space of undotted spinors belonging to the quantum group $SL_q(2,\mathbb{C})$. We mention that $\xi_i^{(l)}$ and $\eta_i^{(l)}$ can be interpreted as Manin quantum planes [6]. For example if we identify $\xi_{-1/2}^{(1/2)} = x$ and $\xi_{1/2}^{(1/2)} = y$ then x and y obey the commutation relation $xy = qyx$. An exact treatment of quantum planes in that framework will be given in section 4.

Using the fact that the comultiplication Δ on $\mathcal{A}_q$ is a $\mathbb{C}$-algebra homomorphism, one can check that $V^{\mathrm{L}}(l)$ ($V^{\mathrm{R}}(l)$) forms a left (right) $\mathcal{A}_q$-subcomodule for each $l \in \frac{1}{2}\mathbb{N}_0$. By direct inspection we find the coproduct (coaction) on the $\xi_i^{(l)}$

$$\Delta\left(\xi_i^{(l)}\right) = \begin{bmatrix} 2l \\ l+i \end{bmatrix}_{q^{-2}}^{1/2} \sum_{k\geqslant 0}^{l-i} \sum_{m\geqslant 0}^{l+i} \begin{bmatrix} 2l \\ m+k \end{bmatrix}_{q^{-2}}^{-1/2} \begin{bmatrix} l-i \\ k \end{bmatrix}_{q^{-2}} \begin{bmatrix} l+i \\ m \end{bmatrix}_{q^{-2}} \times q^{-k(l+i-m)} a^{l-i-k} b^k c^{l+i-m} d^m \otimes \xi_{m+k-l}^{(l)} . \tag{10}$$

Since the $\xi_i^{(l)}$ are linearly independent in $V^{\mathrm{L}}(l)$ matrix elements $w_{i,j}^{(l)} \in \mathcal{A}_q$ are well defined by

$$\Delta\left(\xi_i^{(l)}\right) = \sum_{j\in I_l} w_{i,j}^{(l)} \otimes \xi_j^{(l)} \qquad i \in I_l. \tag{11}$$

They form multiplicative matrices in the sense of Manin [6]. Hence their coproduct and counit are

$$\Delta\left(w_{i,j}^{(l)}\right) = \sum_{k\in I_l} w_{i,k}^{(l)} \otimes w_{k,j}^{(l)} \qquad \epsilon\left(w_{i,j}^{(l)}\right) = \delta_{i,j}. \tag{12}$$

The explicit form of the matrix elements was given in [13, 8]. We have for $-j \leqslant i \leqslant j$

$$w_{i,j}^{(l)} = q^{(i-j)(j-l)} \begin{bmatrix} l-i \\ j-i \end{bmatrix}_{q^{-2}}^{1/2} \begin{bmatrix} l+j \\ j-i \end{bmatrix}_{q^{-2}}^{1/2} P_{l-j}^{(j-i,i+j)}\left(\zeta; q^{-2}\right) b^{j-i} d^{i+j}. \tag{13}$$

The functions $P_n^{(\alpha,\beta)}(z;q)$ are the little q-Jacobi polynomials

$$P_n^{(\alpha,\beta)}(z;q) = \sum_{r\geqslant 0} \frac{\left(q^n; q^{-1}\right)_r \left(q^{-(\alpha+\beta+n+1)}; q^{-1}\right)_r}{\left(q^{-1}; q^{-1}\right)_r \left(q^{-(\alpha+1)}; q^{-1}\right)_r} \left(q^{-1}z\right)^r . \tag{14}$$

The argument of the little q-Jacobi polynomials in (13) is $\zeta = -qbc \in \mathcal{A}_q$. This element plays a crucial role in the representation theory of $SU_q(2)$ since it generates the algebra of all bi-K-invariants in $\mathcal{A}_q$; K corresponds to the subgroup of diagonal matrices of $SL(2)$. Of course, equation (13) does not completely determine all the matrix entries of $w_{i,j}^{(l)}$. However, the missing ones can be obtained by applying the mappings defined in (6) and using $P\left(w_{i,j}^{(l)}\right) = w_{j,i}^{(l)}$ and $Q\left(w_{i,j}^{(l)}\right) = w_{-j,-i}^{(l)}$. For further details we refer the reader to [8, 13].

The explicit form of $w_{i,j}^{(l)}$ and the fact that $V^{\mathrm{L}}(l)$ and $V^{\mathrm{R}}(l)$ are subcomodules of $\mathcal{A}_q$ leads to the following important identifications:

$$w_{i,-l}^{(l)} = \xi_i^{(l)} \qquad w_{-l,j}^{(l)} = \eta_j^{(l)} \qquad i, j \in I_l. \tag{15}$$

It is then easy to obtain the coaction for $V^{\mathrm{R}}(l)$: $\Delta\left(\eta_j^{(l)}\right) = \sum_{i\in I_l} \eta_i^{(l)} \otimes w_{i,j}^{(l)}$.

The matrix elements $w_{i,j}^{(l)}$ naturally carry the $*$-Hopf structure defined on $\mathcal{A}_q$, so in addition to (12) we have

$$w_{i,j}^{(l)*} = (-q)^{j-i}\, w_{-i,-j}^{(l)} \qquad S\left(w_{i,j}^{(l)}\right) = (-q)^{i-j}\, w_{-j,-i}^{(l)} \qquad S\left(w_{i,j}^{(l)}\right)^* = w_{j,i}^{(l)}. \tag{16}$$

Using the restriction of $w^{(l)}_{i,j}$ to the subcomodules $V^{\mathrm{L}}(l)$ and $V^{\mathrm{R}}(l)$ given in (15) shows that $(V^{\mathrm{L}}(l), \Delta)$ and $(V^{\mathrm{R}}(l), \Delta)$ form unitary corepresentations of the quantum group $SU_q(2)$.

It can be shown that the quantum group $SU_q(2)$ admits the decomposition

$$\mathcal{A}_q = \bigoplus_{(i,j,l)\in W} \mathbb{C}w^{(l)}_{i,j} \qquad \text{with } W := \{(i,j,l) \in \tfrac{1}{2}(\mathbb{Z}\times\mathbb{Z}\times\mathbb{N}_0) : i, j \in I_l\} \tag{17}$$

which means that the set $\{w^{(l)}_{i,j} : l \in \frac{1}{2}\mathbb{N}_0,\ i, j \in I_l\}$ forms a $\mathbb{C}$-basis of $\mathcal{A}_q$. The following statements concerning the irreduciblility of $SU_q(2)$ comodules conclude this section.

(i) The comodules $\left(V^{\mathrm{L}}(l), \Delta\right)$ and $\left(V^{\mathrm{R}}(l), \Delta\right)$ are irreducible.
(ii) Every finite-dimensional irreducible $\mathcal{A}_q$-comodule is equivalent to one of the $\mathcal{A}_q$-comodules $\left(V^{\mathrm{L}}(l), \Delta\right)$ and $\left(V^{\mathrm{R}}(l), \Delta\right)$, respectively.
(iii) Every $\mathcal{A}_q$-comodule is completely reducible.

It will be sufficient for us to consider in the following only the comodules $V^{\mathrm{L}}(l)$. Of course, we could equivalently work with $V^{\mathrm{R}}(l)$.

3. Complex conjugate representations of $SL_q(2,\mathbb{C})$

In the representation theory of the Lorentz group one often makes use of the fact that the finite-dimensional representations can be decomposed into two copies of representations of $SU(2)$. These copies belong to the spaces of dotted and undotted spinors of $SL(2,\mathbb{C})$, respectively. As the corepresentations introduced in the preceding section correspond to the space of q-deformed undotted spinors, the construction outlined in this section will lead to q-deformed dotted spinor representations. The analysis here is parallel to that presented above.

Let $\bar{\mathcal{G}} := \mathbb{C}\langle\bar{a}, \bar{b}, \bar{c}, \bar{d}\rangle$ be the associative $\mathbb{C}$-algebra freely generated by $\bar{a}, \bar{b}, \bar{c}, \bar{d}$. $\bar{\mathcal{G}}$ carries a bialgebra structure induced by the matrix

$$\left(\bar{u}^i{}_k\right)_{i,k=1,2} := \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}. \tag{18}$$

$\bar{I}_q$ denotes the biideal generated by relations

$$\begin{aligned} &\bar{a}\bar{b} = q^{-1}\bar{b}\bar{a} && \bar{b}\bar{d} = q^{-1}\bar{d}\,\bar{b} && \bar{b}\bar{c} = \bar{c}\bar{b} \\ &\bar{a}\bar{c} = q^{-1}\bar{c}\bar{a} && \bar{c}\bar{d} = q^{-1}\bar{d}\bar{c} && \bar{a}\bar{d} = \bar{d}\bar{a} - \lambda_q\bar{b}\bar{c}. \end{aligned} \tag{19}$$

Again a unimodularity condition is imposed, $\overline{\det_q}(\bar{u}) = \bar{d}\bar{a} - q\bar{c}\bar{b} = 1$, and added to $\bar{I}_q$. We define $\bar{\mathcal{G}}_q := \bar{\mathcal{G}}/\bar{I}_q$. $\bar{\mathcal{G}}_q$ is a bialgebra with coproduct $\bar{\Delta}$ and counit $\bar{\epsilon}$.

In analogy with [1, 14] we moreover define a multiplicative matrix in $\bar{\mathcal{G}}_q$:

$$\left(v^i{}_j\right)_{i,j=1,2} := \begin{pmatrix} \bar{d} & -q\bar{c} \\ -q^{-1}\bar{b} & \bar{a} \end{pmatrix}. \tag{20}$$

We can now define an algebra homomorphism $j : \mathcal{G} \longrightarrow \bar{\mathcal{G}}$ by $j\left(u^i{}_j\right) := v^i{}_j$. It is easy to show that j induces a bijective morphism of bialgebras $j : \mathcal{G}_q \longrightarrow \bar{\mathcal{G}}_q$. Thus the set

$$\{\bar{d}^p\bar{c}^r\bar{b}^s\bar{a}^t : p = 0 \text{ or } t = 0,\ p, r, s, t \in \mathbb{N}_0\} \tag{21}$$

constitutes a $\mathbb{C}$-basis of $\bar{\mathcal{G}}_q$.

The fact that j is a bijective morphism of bialgebras allows us to define the antipode $\bar{S}$ on $\bar{\mathcal{G}}_q$: $\bar{S} = j \circ S \circ j^{-1}$. Obviously we can identify $\bar{\mathcal{G}}_q$ as a Hopf algebra with $\mathcal{G}_{q^{-1}}$.

Therefore the Hopf algebra $\bar{\mathcal{G}}_q$ naturally carries a $*$-structure $\bar{*} : \bar{\mathcal{G}}_q \longrightarrow \bar{\mathcal{G}}_q$. It is defined in analogy with (4) by $\bar{u}^* := \bar{S}(\bar{u})$.

We define $\bar{\mathcal{A}}_q := (\bar{\mathcal{G}}_q, \bar{\Delta}, \bar{\epsilon}, \bar{S}, \bar{*})$. This is a unitary quantum group naturally isomorphic to $SU_{q^{-1}}(2)$. Moreover $(\mathcal{G}_q, \Delta, \epsilon, S)$ and $(\bar{\mathcal{G}}_q, \bar{\Delta}, \bar{\epsilon}, \bar{S})$ are isomorphic Hopf algebras with respect to the morphism $j : \mathcal{G}_q \longrightarrow \bar{\mathcal{G}}_q$, whereas they are not isomorphic as $*$-Hopf algebras since j does not commute with the $*$-operation.

We now introduce briefly the higher-dimensional representations of $\bar{\mathcal{A}}_q$. The mapping

$$k : \mathcal{A}_q \longrightarrow \bar{\mathcal{A}}_q \qquad (u^i{}_j) \longmapsto (\bar{u}^i{}_j) \tag{22}$$

defines a natural bijective conjugate linear algebra antihomomorphism. We have

$$\bar{\Delta} \circ k = (k \otimes k) \circ \Delta \qquad {}^{-} \circ \epsilon = \bar{\epsilon} \circ k. \tag{23}$$

Define $\bar{w}^{(l)}_{i,j} := k(w^{(l)}_{i,j}) \in \bar{\mathcal{A}}_q$. Of course, $\bar{w}^{(l)}_{i,j}$ is again a multiplicative matrix in $\bar{\mathcal{A}}_q$ and because of

$$j\left(w^{(l)}_{i,j}\right) = (-q)^{j-i}\, \bar{w}^{(l)}_{-i,-j} \tag{24}$$

together with (17) and (21) it is clear that the set

$$\{\bar{w}^{(l)}_{i,j} : i, j \in I_l,\ l \in \tfrac{1}{2}\mathbb{N}_0\} \tag{25}$$

constitutes a $\mathbb{C}$-basis of $\bar{\mathcal{A}}_q$.

The left corepresentation spaces of $\bar{\mathcal{A}}_q$ are obtained by using the properties of the mapping k. We define $\mathbb{C}$-vector subspaces $\bar{V}^{\mathrm{L}}(l) \subset \bar{\mathcal{A}}_q$, $l \in \frac{1}{2}\mathbb{N}_0$, by

$$\bar{V}^{\mathrm{L}}(l) := \bigoplus_{i \in I_l} \mathbb{C}\bar{\xi}^{(l)}_i \qquad \bar{\xi}^{(l)}_i := k\left(\xi^{(l)}_i\right) = \begin{bmatrix} 2l \\ l+i \end{bmatrix}^{1/2}_{q^{-2}} \bar{c}^{l+i}\bar{a}^{l-i}. \tag{26}$$

From (22) we can deduce the coaction $\bar{\Delta}\left(\bar{\xi}^{(l)}_i\right) = \sum_{j \in I_l} \bar{w}^{(l)}_{i,j} \otimes \bar{\xi}^{(l)}_j$.

We call the $\bar{\mathcal{A}}_q$-comodules $\left(\bar{V}^{\mathrm{L}}(l), \bar{\Delta}\right)$ complex conjugate corepresentations of the quantum group $SL_q(2, \mathbb{C})$ belonging to the spin l. We remark that the corepresentation spaces $\left(\bar{V}^{\mathrm{L}}(l), \bar{\Delta}\right)$ are unitary since $\bar{S}\left(\bar{w}^{(l)}_{i,j}\right)^{\bar{*}} = \bar{w}^{(l)}_{j,i}$ in $\bar{\mathcal{A}}_q$. All other statements concerning the representations given in section 2 directly apply to the complex conjugate representations using the mapping k.

4. Representations of the quantum Lorentz group

Let $\mathcal{C}_q := \mathcal{A}_q \otimes \bar{\mathcal{A}}_q$. As a tensor product of the Hopf algebras $\mathcal{A}_q$ and $\bar{\mathcal{A}}_q$, $\mathcal{C}_q$ becomes a Hopf algebra with coproduct $\Delta^{\otimes} = \left(\mathrm{id}_{\mathcal{A}_q} \otimes \tau \otimes \mathrm{id}_{\bar{\mathcal{A}}_q}\right) \circ \left(\Delta \otimes \bar{\Delta}\right)$, counit $\epsilon^{\otimes} = \epsilon \otimes \bar{\epsilon}$ and antipode $S^{\otimes} = S \otimes \bar{S}$. Here τ denotes the flip automorphism. The mapping $k : \mathcal{A}_q \longrightarrow \bar{\mathcal{A}}_q$ which was introduced in the previous section induces an involution ${}^{-} : \mathcal{C}_q \longrightarrow \mathcal{C}_q$ via $\overline{a \otimes b} := k^{-1}(b) \otimes k(a)$ for all $a \in \mathcal{A}_q$ and $b \in \bar{\mathcal{A}}_q$. It is possible to define a second involution $*^{\otimes}$ on $\mathcal{C}_q$ which is defined by $*^{\otimes} := * \otimes \bar{*}$.

It follows that $(\mathcal{C}_q, \Delta^{\otimes}, \epsilon^{\otimes}, S^{\otimes}, -)$ and $(\mathcal{C}_q, \Delta^{\otimes}, \epsilon^{\otimes}, S^{\otimes}, *^{\otimes})$ are $*$-Hopf algebras. The first one leads to unphysical representations, as was pointed out in [1]. From the second one representations of $SO_q(4)$ can be obtained by applying the Hopf algebra morphism $\mathrm{id} \otimes j^{-1}$. Without a reality condition this Hopf algebra leads to the quantum group $SL_q(2, \mathbb{C})$ [2].

We denote by $\mathcal{L} := \mathbb{C}\langle a, b, c, d, \bar{a}, \bar{b}, \bar{c}, \bar{d}\rangle$ a free associative algebra. $\mathcal{G}$ and $\bar{\mathcal{G}}$ can obviously be identified as subalgebras of $\mathcal{L}$. $\mathcal{L}$ carries the structure of a bialgebra induced by the matrix

$$\left(m^i{}_j\right)_{i,j=1,\dots,4} := \begin{pmatrix} \left(u^k{}_l\right)_{k,l=1,2} & 0 \\ 0 & \left(\bar{u}^k{}_l\right)_{k,l=1,2} \end{pmatrix}. \tag{27}$$

J_q is the biideal in $\mathcal{L}$ which is generated by relations (3), (19) and [1, 2, 14]:

$$\begin{array}{ll} a\bar{a} = \bar{a}a - q\lambda_q\bar{c}c & c\bar{a} = q\bar{a}c \\ a\bar{b} = q^{-1}\bar{b}a - \lambda_q\bar{d}c & c\bar{b} = \bar{b}c \\ a\bar{c} = q\bar{c}a & c\bar{c} = \bar{c}c \\ a\bar{d} = \bar{d}a & c\bar{d} = q^{-1}\bar{d}c \\ b\bar{a} = q^{-1}\bar{a}b - \lambda_q\bar{c}d & d\bar{a} = \bar{a}d \\ b\bar{b} = \bar{b}b + q\lambda_q\left(a\bar{a} - \bar{d}d\right) & d\bar{b} = q\bar{b}d + q\lambda_q c\bar{a} \\ b\bar{c} = \bar{c}b & d\bar{c} = q^{-1}\bar{c}d \\ b\bar{d} = q\bar{d}b + q\lambda_q a\bar{c} & d\bar{d} = \bar{d}d + q\lambda_q\bar{c}c. \end{array} \tag{28}$$

Again we add to J_q the unimodularity conditions of the submatrices u and $\bar{u}$ which makes the matrix (27) itself unimodular. Let $\mathcal{L}_q := \mathcal{L}/J_q$. A conjugation on $\mathcal{L}_q$ is defined by $^- : \mathcal{L}_q \longrightarrow \mathcal{L}_q$, $u^i{}_j \longmapsto \bar{u}^i{}_j$, $\bar{u}^i{}_j \longmapsto u^i{}_j$. An antipode on $\mathcal{L}_q$ is given via the antipodal maps on the matrices u and $\bar{u}$. We call the $*$-Hopf algebra $\mathcal{L}_q$ the quantum group $SL_q(2,\mathbb{C})$. Using the diamond lemma [15] one can show that the set of monomials

$$\{a^i b^j c^k d^l \bar{d}^p \bar{c}^r \bar{b}^s \bar{a}^t \quad i = 0 \text{ or } l = 0,\, p = 0 \text{ or } t = 0\} \tag{29}$$

with $i, j, k, l, p, r, s, t \in \mathbb{N}_0$ constitutes a $\mathbb{C}$-basis of $\mathcal{L}_q$. This shows that $\mathcal{A}_q$ and $\bar{\mathcal{A}}_q$ form sub-Hopf algebras of $\mathcal{L}_q$. We now define a $\mathbb{C}$-linear mapping which turns out to be important in the representation theory of the q-deformed Lorentz algebra

$$r : \mathcal{A}_q \otimes \bar{\mathcal{A}}_q \longrightarrow \mathcal{L}_q : \quad a^i b^j c^k d^l \otimes \bar{d}^p \bar{c}^r \bar{b}^s \bar{a}^t \longmapsto a^i b^j c^k d^l \bar{d}^p \bar{c}^r \bar{b}^s \bar{a}^t. \tag{30}$$

This mapping is well defined. Note that using (17) and (21) the set $\{a^i b^j c^k d^l \otimes \bar{d}^p \bar{c}^r \bar{b}^s \bar{a}^t :$ $i, j, k, l, p, r, s, t \in \mathbb{N}_0,\ i = 0 \text{ or } l = 0,\ p = 0 \text{ or } t = 0\}$ is a $\mathbb{C}$-basis of $\mathcal{A}_q \otimes \bar{\mathcal{A}}_q$. Since the set (29) forms a $\mathbb{C}$-basis of $\mathcal{L}_q$, we have that r is an isomorphism and it follows that

$$r\left(w^{(l_1)}_{i_1,j_1} \otimes \bar{w}^{(l_2)}_{i_2,j_2}\right) = w^{(l_1)}_{i_1,j_1}\bar{w}^{(l_2)}_{i_2,j_2} \qquad \text{for all } l_k \in \mathbb{N}_0 \quad i_k, j_k \in I_{l_k} \quad k = 1, 2. \tag{31}$$

This is equivalent to the statement that

$$\left\{w^{(l_1)}_{i_1,j_1}\bar{w}^{(l_2)}_{i_2,j_2} : l_1, l_2 \in \mathbb{N}_0,\ i_1, j_1 \in I_{l_1},\ i_2, j_2 \in I_{l_2}\right\}$$

is a $\mathbb{C}$-basis of $\mathcal{L}_q$. In other words we can say that the representations of $SL_q(2,\mathbb{C})$ can, as in the classical case, be characterized by a pair (l_1, l_2) which denotes the highest weights of the representations of $\mathcal{A}_q$ and $\bar{\mathcal{A}}_q$, respectively.

It is important to note that the mapping $r : \mathcal{C}_q \longrightarrow \mathcal{L}_q$ is a morphism of coalgebras which can be shown by direct inspection on the basis of $\mathcal{A}_q \otimes \bar{\mathcal{A}}_q$.

Now we can introduce the left corepresentation spaces $V^{\mathrm{L}}(l_1, l_2) \subset \mathcal{L}_q$, $l_1, l_2 \in \frac{1}{2}\mathbb{N}_0$ of the QLGr by

$$V^{\mathrm{L}}(l_1, l_2) := \bigoplus_{\substack{i_1 \in I_{l_1} \\ i_2 \in I_{l_2}}} \mathbb{C}\xi^{(l_1,l_2)}_{i_1,i_2} \qquad \xi^{(l_1,l_2)}_{i_1,i_2} := \xi^{(l_1)}_{i_1}\bar{\xi}^{(l_2)}_{i_2}. \tag{32}$$

By definition, we get the coaction

$$\Delta\left(\xi^{(l_1,l_2)}_{i_1,i_2}\right) = \sum_{j_1 \in I_{l_1}}\sum_{j_2 \in I_{l_2}} w^{(l_1)}_{i_1,j_1}\bar{w}^{(l_2)}_{i_2,j_2} \otimes \xi^{(l_1,l_2)}_{j_1,j_2}. \tag{33}$$

The $\mathcal{L}_q$-comodules $\left(V^{\mathrm{L}}(l_1, l_2), \Delta\right)$ are corepresentations of the quantum group $SL_q(2, \mathbb{C})$. Their irreducibility follows from our previous results.

The analysis shows that the comodules (32) provide us with a q-deformed spinor calculus for arbitrary finite-dimensional representations of $SL_q(2, \mathbb{C})$. To make contact with the usual $SL(2, \mathbb{C})$ spinor calculus, see e.g. [16], we write symbolically

$$z^{\alpha}_{(l)} \sim \xi^{(l)}_i \qquad z_{(l)\,\alpha} \sim k \circ j\left(\xi^{(l)}_i\right) \qquad \bar{z}^{\dot{\alpha}}_{(l)} \sim k\left(\xi^{(l)}_i\right) \qquad \bar{z}_{(l)\,\dot{\alpha}} \sim j\left(\xi^{(l)}_i\right). \tag{34}$$

In [7] the concept of complex quantum planes in generalization of Manin's [6] construction was introduced. We will now show how the two-dimensional complex quantum plane can be understood in the framework of $SL_q(2, \mathbb{C})$ corepresentations.

Let $\mathcal{S}_q$ be the associative $\mathbb{C}$-algebra generated by elements $x, y, \bar{x}, \bar{y}$ and relations

$$\begin{aligned} xy &= qyx \qquad & x\bar{y} &= q\bar{y}x \qquad & y\bar{y} &= \bar{y}y \\ \bar{y}\bar{x} &= q\bar{x}\bar{y} \qquad & y\bar{x} &= q\bar{x}y \qquad & x\bar{x} &= \bar{x}x - q\lambda_q \bar{y}y. \end{aligned} \tag{35}$$

$\mathcal{S}_q$ obviously carries a conjugation

$$- : \mathcal{S}_q \longrightarrow \mathcal{S}_q \qquad x \longmapsto \bar{x}, \quad y \longmapsto \bar{y}. \tag{36}$$

The algebra $(\mathcal{S}_q, -)$ is the complex two-dimensional quantum plane of [7]. With the help of the diamond lemma one can proof that $\{x^i y^j \bar{y}^k \bar{x}^l : i, j, k, l \in \mathbb{N}_0\}$ is a $\mathbb{C}$-basis of $\mathcal{S}_q$.

We define a $*$-algebra morphism $\mathcal{J} : \mathcal{S}_q \longrightarrow \mathcal{L}_q$ by $\mathcal{J}(x) := a$, $\mathcal{J}(y) := c$. $\mathcal{J}$ is well defined because $a, c, \bar{a}, \bar{c} \in \mathcal{L}_q$ obey the same commutation relations as $x, y, \bar{x}, \bar{y} \in \mathcal{S}_q$. Obviously $\mathcal{J}$ is injective $\mathcal{J}\left(x^i y^j \bar{y}^k \bar{x}^l\right) = a^i c^j \bar{c}^k \bar{a}^l$ with $i, j, k, l \in \mathbb{N}_0$. Via the identification $\mathcal{J}$ the quantum plane $\mathcal{S}_q$ can be considered as a subalgebra of $\mathcal{L}_q$. If we define for $\nu \in \mathbb{N}_0$:

$$S_\nu := \operatorname{span}_{\mathbb{C}} \left\{\xi^{(l_1, l_2)}_{i_1, i_2} : l_1 + l_2 = \tfrac{1}{2}\nu,\ i_k \in I_{l_k},\ l_k \in \tfrac{1}{2}\mathbb{N}_0,\ k = 1, 2\right\} \tag{37}$$

we then have

$$\mathcal{J}\left(\mathcal{S}_q\right) = \bigoplus_{\nu \in \mathbb{N}_0} S_\nu\,. \tag{38}$$

This implies that $\mathcal{S}_q$ is a quadratic algebra. Thus by (37) the quantum plane can, roughly speaking, be indentified with the corepresentation spaces $V^{\mathrm{L}}(l_1, l_2)$ of (32). We remark that the 'second spinor copy' used in [1,3] to obtain a q-Minkowski space with non-vanishing length is just given by $\mathcal{Q}\left(V^{\mathrm{R}}\right) \mathcal{Q}\left(k(V^{\mathrm{R}})\right)$ with the help of the right-comodules and the properties of the mapping $\mathcal{Q}$ (6). Note that this comodule does not form a subalgebra of (28).

5. Differential representations

Given a Hopf algebra $(H, \nabla, \eta, \Delta, \epsilon, S)$ with ∇ the multiplication and η the unit map, we denote by $(H^\circ, \nabla^\circ, \eta^\circ, \Delta^\circ, \epsilon^\circ, S^\circ)$ the maximal Hopf algebra in the algebraic dual H^* of H which is induced by the structure maps of H [10]. For example, the coproduct Δ° of H° is defined as $\Delta^\circ(\varphi) = \varphi \circ \nabla$ for $\varphi \in H^\circ$. H° can be considered as a Hopf algebra of regular functionals acting on H.

If $(H, *)$ is a $*$-Hopf algebra then there are two possibilities for a $*$-structure in H°. We have, for $\varphi \in H^\circ$,

$$\begin{aligned} *^\circ : H^\circ \longrightarrow H^\circ \qquad & \varphi \longmapsto \overline{\varphi} \circ * \circ S \\ \star^\circ : H^\circ \longrightarrow H^\circ \qquad & \varphi \longmapsto \overline{\varphi} \circ * \circ S^{-1}. \end{aligned} \tag{39}$$

It will turn out that the second $*$-structure mentioned will lead to a second possibility to introduce a conjugation on the generators of the q-deformed Lorentz algebra different from the one developed in [3] as long as $q \neq 1$.

Having the concept of the dual Hopf algebra we can define representations of the dual Hopf algebra. Given a Hopf algebra (H, Δ, ϵ, S) and a left (right) H-comodule (V, δ) we can define $\mathbb{C}$-linear mappings $\rho : H^* \longrightarrow \mathrm{End}_{\mathbb{C}}(V)$ and $\lambda : H^* \longrightarrow \mathrm{End}_{\mathbb{C}}(V)$ by

$$\rho(\varphi) := (\varphi \otimes \mathrm{id}_V) \circ \delta \qquad \text{and} \qquad \lambda(\varphi) := (\mathrm{id}_V \otimes \varphi) \circ \delta. \tag{40}$$

We use the abbreviations $\widehat{\varphi} := \rho(\varphi)$ and $\widetilde{\varphi} := \lambda(\varphi)$. For our purposes (V, δ) is a subcomodule of (H, Δ). It turns out that if (V, δ) is a left-H subcomodule then (ρ, V) is an anti-representation and if it is a right-H subcomodule then (λ, V) is a representation of H^* (H^0) on V. As mentioned above, we will consider only left-H-subcomodules and are therefore naturally led to anti-representations.

For the finite-dimensional left-H-comodule (V, δ) we denote a scalar product by $\langle \cdot \mid \cdot \rangle_{\mathrm{L}}$. Let us define the mapping

$$(\cdot, \cdot)_{\mathrm{L}} : (H \otimes V) \times (H \otimes V) \longrightarrow H : \qquad (a \otimes \xi) \times (b \otimes \eta) \longmapsto ab^* \langle \xi \mid \eta \rangle_{\mathrm{L}}. \tag{41}$$

The existence of the scalar product in our case is guaranteed since we are dealing with comodules of the quantum group $SU_q(2)$. Here the scalar product is given by the Haar measure, see e.g. [11, 17]. This also gives us the concept of a Hilbert space structure on the comodules. Their unitarity can be defined in this way [8]. However, this scalar product is not bi-invariant with respect to $SL_q(2, \mathbb{C})$. When comparing our results with those of [3] one has to notice that the scalar product (41) is conjugate linear in the second component.

We now outline the construction of the q-deformed dual Hopf algebra of $\mathcal{A}_q$. This is, in principle, well known from [18, 19, 8]. However, this serves as a model for introducing the non-compact part of the q-deformed Lorentz algebra. We define on the generators a, b, c, d

$$\begin{aligned} k^{\pm}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &:= \begin{pmatrix} q^{\mp 1/2} & 0 \\ 0 & q^{\pm 1/2} \end{pmatrix} \\ e\begin{pmatrix} a & b \\ c & d \end{pmatrix} &:= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ f\begin{pmatrix} a & b \\ c & d \end{pmatrix} &:= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{42}$$

These actions have to be continued on arbitrary monomials of $\mathcal{G}_q$ as defined in section 2. The following continuations make the mappings $\check{k}^{\pm}, \check{e}, \check{f} \in \mathcal{G}^*$ well defined:

$$\begin{aligned} &\check{k}^{\pm}(ab) = \check{k}^{\pm}(a)\check{k}^{\pm}(b) && \check{k}^{\pm}(1) = 1 \\ &\check{e}(ab) = \check{e}(a)\check{k}^{+}(b) + \check{k}^{-}(a)\check{e}(b) && \check{e}(1) = 0 \\ &\check{f}(ab) = \check{f}(a)\check{k}^{+}(b) + \check{k}^{-}(a)\check{f}(b) && \check{f}(1) = 0 \end{aligned} \tag{43}$$

where a and b denote arbitrary elements of $\mathcal{G}$. Let $\left(\mathcal{A}_q^\circ, \Delta^\circ, \epsilon^\circ, S^\circ\right)$ denote the maximal Hopf algebra induced by $\left(\mathcal{A}_q, \Delta, \epsilon, S\right)$. From (43) by building the quotient one obtains mappings $k^{\pm}, e, f \in \mathcal{A}_q^\circ$ whose comultiplications and counits are given by

$$\begin{aligned} &\Delta^\circ\left(k^{\pm}\right) = k^{\pm} \otimes k^{\pm} && \epsilon^\circ\left(k^{\pm}\right) = 1 \\ &\Delta^\circ(e) = e \otimes k^{+} + k^{-} \otimes e && \epsilon^\circ(e) = 0 \\ &\Delta^\circ(f) = f \otimes k^{+} + k^{-} \otimes f && \epsilon^\circ(f) = 0. \end{aligned} \tag{44}$$

The duality structure implies the well known algebraic relations

$$\begin{aligned} k^+k^- &= k^-k^+ = 1^\circ \qquad & k^+ek^- &= q^{-1}e \\ fe - ef &= \frac{k^{+2} - k^{-2}}{q - q^{-1}} \qquad & k^+fk^- &= qf. \end{aligned} \tag{45}$$

We define $\mathcal{U}_q := \mathcal{U}_q su(2) = \mathcal{U}\mathbb{C}\langle k^+, k^-, e, f\rangle$. $\mathcal{U}_q$ is the deformed universal enveloping Hopf algebra of $su(2)$. It has the antipode S°:

$$S^\circ\left(k^\pm\right) = k^\mp \qquad S^\circ(e) = -q^{-1}e \qquad S^\circ(f) = -qf. \tag{46}$$

We mention that the algebra $\mathcal{U}_q$ has the $\mathbb{C}$-basis $\{e^\mu k^\lambda f^\nu : \mu, \nu \in \mathbb{N}_0,\ \lambda \in \mathbb{Z}\}$. Following (39) we have two possible $*$-structures on the generators of $\mathcal{U}_q$:

$$\begin{aligned} \left(k^\pm\right)^* &= k^\pm \qquad & e^* &= f \qquad & f^* &= e \\ \left(k^\pm\right)^\star &= k^\pm \qquad & e^\star &= q^{-2}f \qquad & f^\star &= q^2 e. \end{aligned} \tag{47}$$

We now want to find Hilbert space representations of the generators of $\mathcal{U}_q$. Hence in a first step we calculate using (44) the actions of the generators on the basis elements $w_{i,j}^{(l)}$ of $\mathcal{A}_q$.

$$k^\pm\left(w_{i,j}^{(l)}\right) = q^{\pm i}\delta_{i,j} \qquad e\left(w_{i,j}^{(l)}\right) = q^{l-\frac{1}{2}}E_q(l,i)\delta_{i+1,j} \qquad f\left(w_{i,j}^{(l)}\right) = q^{l-\frac{1}{2}}E_q(l,-i)\delta_{i-1,j}. \tag{48}$$

The abbreviation for $l \in \frac{1}{2}\mathbb{N}_0$, $i \in I_l$ is used:

$$E_q(l,i) := \left([l-i]_{q^{-2}}[l+i+1]_{q^{-2}}\right)^{1/2}. \tag{49}$$

The Hilbert space (anti-) representations are found if we restrict the actions on $w_{i,j}^{(l)}$ to the subcomodules $V^{\mathrm{L}}(l)$ as outlined in (15):

$$\widehat{k^\pm}\left(\xi_i^{(l)}\right) = q^{\pm i}\xi_i^{(l)} \qquad \widehat{e}\left(\xi_i^{(l)}\right) = q^{l-\frac{1}{2}}E_q(l,i)\xi_{i+1}^{(l)} \qquad \widehat{f}\left(\xi_i^{(l)}\right) = q^{l-\frac{1}{2}}E_q(l,-i)\xi_{i-1}^{(l)}. \tag{50}$$

$V^{\mathrm{L}}(l)$ becomes a Hilbert space if we define the sets $\{\xi_i^{(l)} : i \in I_l\}$ to be orthogonal with respect to $\langle\cdot\mid\cdot\rangle_{\mathrm{L}}$. It follows that $\left(\rho, \left(V^{\mathrm{L}}(l), \langle\cdot\mid\cdot\rangle_{\mathrm{L}}\right)\right)$ is a unitary antirepresentation of $\left(\mathcal{A}_q^\circ, *\right)$.

Analogous statements can be given for $V^{\mathrm{R}}(l)$. In this case one deals with a representation rather than an antirepresentation and has to use a slightly different scalar product.

To conclude we say a few words about the Casimir element in $\mathcal{U}_q$. It is defined by the mapping $C \in \mathcal{A}_q^\circ$ via

$$C := \frac{q^{-1}k^{+2} + qk^{-2} - q - q^{-1}}{\left(q - q^{-1}\right)^2} + fe. \tag{51}$$

A straightforward calculation gives its matrix elements

$$C\left(w_{i,j}^{(l)}\right) = q[l]_{q^2}[l+1]_{q^{-2}}\delta_{i,j}. \tag{52}$$

The Casimir has the property to be real under both possible $*$-structures, i.e. $C^* = C$ and $C^\star = C$. The meaning of this invariance becomes obvious by the following remark. The $*$-Hopf algebras $\left(\mathcal{A}_q^\circ, *\right)$ and $\left(\bar{\mathcal{A}}_q^\circ, \star\right)$ are isomorphic—apply the mapping $\left(j^{-1}\right)_\circ^* : \mathcal{A}_q^\circ \longrightarrow \bar{\mathcal{A}}_q^\circ$.

6. q-deformed Lorentz algebra

In this section we will prove that the q-deformed Lorentz algebra of [3] can be understood via differential representation as an algebra dual to the quantum group $SL_q(2,\mathbb{C})$. We will make use of the formalism outlined in the preceding section. In what follows $\left(\bar{\mathcal{A}}_q^\circ, \Delta^\circ, \epsilon^\circ, S^\circ\right)$ and $\left(\mathcal{L}_q^\circ, \Delta^\circ, \epsilon^\circ, S^\circ\right)$ denote the maximal dual Hopf algebras induced by $\left(\bar{\mathcal{A}}_q, \Delta, \epsilon, S\right)$ and $\left(\mathcal{L}_q, \Delta, \epsilon, S\right)$, respectively.

We introduce the differential representation $\rho : \mathcal{L}_q^* \longrightarrow \mathrm{End}_{\mathbb{C}}\left(\mathcal{L}_q\right)$, as in the previous section. On the basis $w_{i_1,j_1}^{(l_1)}\bar{w}_{i_2,j_2}^{(l_2)} \in \mathcal{L}_q$ the action is given by

$$\widehat{\varphi}\left(w_{i_1,j_1}^{(l_1)}\bar{w}_{i_2,j_2}^{(l_2)}\right) = \sum_{k_1\in I_{l_1}}\sum_{k_2\in I_{l_2}} \varphi\left(w_{i_1,k_1}^{(l_1)}\bar{w}_{i_2,k_2}^{(l_2)}\right) w_{k_1,j_1}^{(l_1)}\bar{w}_{k_2,j_2}^{(l_2)} \tag{53}$$

with $\left(\varphi \in \mathcal{L}_q^*\right)$. We restrict ourselves to the $\mathcal{L}_q$-subcomodule $\left(V^{\mathrm{L}}(l_1,l_2), \Delta\right)$ following (15):

$$\widehat{\varphi}\left(\xi_{i_1,i_2}^{(l_1,l_2)}\right) = \sum_{j_1\in I_{l_1}}\sum_{j_2\in I_{l_2}} \varphi\left(w_{i_1,j_1}^{(l_1)}\bar{w}_{i_2,j_2}^{(l_2)}\right)\xi_{j_1,j_2}^{(l_1,l_2)}. \tag{54}$$

For the quantum plane introduced in section 4 we have the following important theorem.

Theorem 1. The antirepresentation $\left(\rho, \mathcal{J}\left(\mathcal{S}_q\right)\right)$ of $\mathcal{L}_q^*$ is faithful.

Proof. The set $\left\{w_{i_1,j_1}^{(l_1)}\bar{w}_{i_2,j_2}^{(l_2)} : l_1, l_2 \in \mathbb{N}_0,\ i_1, j_1 \in I_{l_1},\ i_2, j_2 \in I_{l_2}\right\}$ is, following (31), a $\mathbb{C}$-basis of $\mathcal{L}_q$. With $\mathcal{J}\left(\mathcal{S}_q\right) = \bigoplus_{l_1,l_2\in\frac{1}{2}\mathbb{N}_0} V^{\mathrm{L}}(l_1,l_2)$ and (54) follows the claim of the theorem.

As in section 5 the comodules $\bar{V}^{\mathrm{L}}(l)$ admit the structure of a Hilbert space if the sets $\{\bar{\xi}_i^{(l)} : i \in I_l\}$ are defined to be orthogonal with respect to a scalar product $\langle\cdot \mid \cdot\rangle_{\mathrm{L}}$. This makes the subcomodules into unitary antirepresentations of $\left(\bar{\mathcal{A}}_q^\circ, \bar{*}\right)$.

We are now able to define the compact part of the q-deformed Lorentz algebra. We define an algebra homomorphism $\check{p} : \mathcal{L} \longrightarrow \mathcal{G} \subset \mathcal{L}$ by

$$\check{p}\begin{pmatrix} \left(u^i{}_k\right) & 0 \\ 0 & \left(v^i{}_k\right) \end{pmatrix} := \begin{pmatrix} \left(u^i{}_k\right) & 0 \\ 0 & \left(u^i{}_k\right) \end{pmatrix}. \tag{55}$$

It is not difficult to show that $\check{p}$ induces an algebra homomorphism $p : \mathcal{L}_q \longrightarrow \mathcal{A}_q \subset \mathcal{L}_q$. Moreover we have that the mapping

$$p : \left(\mathcal{L}_q, \Delta, \epsilon, S, -\right) \longrightarrow \left(\mathcal{A}_q, \Delta, \epsilon, S, *\right) \tag{56}$$

is a $*$-Hopf algebra morphism. These statements can be lifted to the dual Hopf algebras. This means that the mapping

$$p_\circ^* : \mathcal{A}_q^\circ \longrightarrow \mathcal{L}_q^\circ \tag{57}$$

is a $*$-Hopf algebra morphism with respect to both star structures $*$ and $\star$ (equation (39)) on $\mathcal{A}_q^\circ$ and $\mathcal{L}_q^\circ$.

This has the important consequence that there are continuations $e_{\mathcal{L}}, f_{\mathcal{L}}, k_{\mathcal{L}}^\pm \in \mathcal{L}_q^\circ$ of the generators $e, f, k^\pm \in \mathcal{A}_q^\circ$ such that $e_{\mathcal{L}}, f_{\mathcal{L}}, k_{\mathcal{L}}^\pm$ posess the same coproducts, counits, antipodes, star structures and have the same algebraic relations as $e, f, k^\pm$ in $\mathcal{A}_q^\circ$.

So far we have worked with the Drinfeld–Jimbo basis in $\mathcal{U}_q$. Since we want to make contact with [3] we define

$$(\tau^3)^{\pm 1/2} := k_{\mathcal{L}}^{\mp 2} \qquad T^+ := q^{1/2} e_{\mathcal{L}} k_{\mathcal{L}}^- \qquad T^- := q^{-1/2} f_{\mathcal{L}} k_{\mathcal{L}}^-. \tag{58}$$

This shows that $T^{\pm}$, T^3, $(\tau^3)^{1/2}$ and $(\tau^3)^{-1/2}$ are elements of $\mathcal{L}_q^\circ$ and their algebra is

$$q^{-1}T^-T^+ - qT^+T^- = \lambda_q^{-1}(1-\tau^3) \qquad T^+\tau^3 = q^{-2}\tau^3T^+ \qquad T^-\tau^3 = q^2\tau^3T^-. \tag{59}$$

The Hopf-$*$-structure of these generators is easily obtained from those of $e, f, k^{\pm}$ given above. We mention that in [3] only the complex structure '$*$' in (47) has been recognized.

By direct calculation using the comultiplication rules we find the actions on the basis $w^{(l_1)}_{i_1,j_1}\bar{w}^{(l_2)}_{i_2,j_2} \in \mathcal{L}_q$:

$$\begin{aligned}
T^+\big(w^{(l_1)}_{i_1,j_1}\bar{w}^{(l_2)}_{i_2,j_2}\big) &= q^{l_1-i_1-1}E_q\,(l_1,i_1)\,\delta_{j_1,i_1+1}\delta_{j_2,i_2} - q^{-2i_1-2}q^{l_2+i_2}E_q\,(l_2,-i_2)\,\delta_{j_1,i_1}\delta_{j_2,i_2-1}\\
T^-\big(w^{(l_1)}_{i_1,j_1}\bar{w}^{(l_2)}_{i_2,j_2}\big) &= q^{l_1-i_1}E_q\,(l_1,-i_1)\,\delta_{j_1,i_1-1}\delta_{j_2,i_2} - q^{-2i_1+1}q^{l_2+i_2}E_q\,(l_2,i_2)\,\delta_{j_1,i_1}\delta_{j_2,i_2+1}\\
\big(\tau^3\big)^{1/2}\big(w^{(l_1)}_{i_1,j_1}\bar{w}^{(l_2)}_{i_2,j_2}\big) &= q^{-2i_1+2i_2}\delta_{j_1,i_1}\delta_{j_2,i_2}.
\end{aligned} \tag{60}$$

Restricting ourselves as usual to the subcomodules leads to the antirepresentations of these generators on the comodules $V^{\mathrm{L}}(l)$ and $\bar{V}^{\mathrm{L}}(l)$.

Up to now it has not quite been understood whether the q-deformed Lorentz algebra [3] could be regarded as the $*$-Hopf algebra dual to $SL_q(2,\mathbb{C})$. This is because the algebra was found by studying only the actions of the generators on the elements of the complex quantum plane. It appears at first that the algebra seems to have seven generators and one degree of freedom is removed by heuristically finding a central quantity in that algebra. We think that these uncertainties justify our detailed analysis.

By the above considerations we have already treated completely the compact part of the q-deformed Lorentz algebra of [3]. We now introduce generators $\tau_a, \sigma_a, T_a, S_a \in \mathcal{L}$ and define their actions on the fundamental representations of $SL_q(2,\mathbb{C})$ by

$$\begin{aligned}
&\tau_a\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} q^{-1/2} & 0\\ 0 & q^{1/2}\end{pmatrix} \qquad && \tau_a\begin{pmatrix} \bar{a} & \bar{b}\\ \bar{c} & \bar{d}\end{pmatrix} = \begin{pmatrix} q^{-1/2} & 0\\ 0 & q^{1/2}\end{pmatrix}\\
&\sigma_a\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} q^{1/2} & 0\\ 0 & q^{-1/2}\end{pmatrix} && \sigma_a\begin{pmatrix} \bar{a} & \bar{b}\\ \bar{c} & \bar{d}\end{pmatrix} = \begin{pmatrix} q^{1/2} & 0\\ 0 & q^{-1/2}\end{pmatrix}\\
&T_a\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} 0 & q^{1/2}\\ 0 & 0\end{pmatrix} && T_a\begin{pmatrix} \bar{a} & \bar{b}\\ \bar{c} & \bar{d}\end{pmatrix} = \begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}\\
&S_a\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix} && S_a\begin{pmatrix} \bar{a} & \bar{b}\\ \bar{c} & \bar{d}\end{pmatrix} = \begin{pmatrix} 0 & q^{1/2}\\ 0 & 0\end{pmatrix}.
\end{aligned} \tag{61}$$

The key point of our proof is to find proper continuations on $\mathcal{L}$ such that the mappings $\check{\tau}_a$, $\check{\sigma}_a$, $\check{T}_a$, $\check{S}_a \in \mathcal{L}^*$ become well defined. For $a, b \in \mathcal{L}$ this is achieved by

$$\begin{aligned}
&\check{\tau}_a\,(ab) = \check{\tau}_a\,(a)\,\check{\tau}_a\,(b) + \lambda_q^2\check{S}_a\,(a)\,\check{T}_a\,(b) \qquad && \check{\tau}_a\,(1) = 1\\
&\check{\sigma}_a\,(ab) = \check{\sigma}_a\,(a)\,\check{\sigma}_a\,(b) + \lambda_q^2\check{T}_a\,(a)\,\check{S}_a\,(b) && \check{\sigma}_a\,(1) = 1\\
&\check{T}_a\,(ab) = \check{T}_a\,(a)\,\check{\tau}_a\,(b) + \check{\sigma}_a\,(a)\,\check{T}_a\,(b) && \check{T}_a\,(1) = 0\\
&\check{S}_a\,(ab) = \check{S}_a\,(a)\,\check{\sigma}_a\,(b) + \check{\tau}_a\,(a)\,\check{S}_a\,(b) && \check{S}_a\,(1) = 0.
\end{aligned} \tag{62}$$

From $\check{\tau}_a, \check{\sigma}_a, \check{T}_a, \check{S}_a \in \mathcal{L}^*$ by forming the quotient one obtains the mappings $\tau_a, \sigma_a, T_a, S_a \in$

$\mathcal{L}_q^\circ$, such that the following relations hold:

$$\begin{aligned} &\Delta^\circ(\tau_a) = \tau_a \otimes \tau_a + \lambda_q^2 S_a \otimes T_a && \epsilon^\circ(\tau_a) = 1 \\ &\Delta^\circ(\sigma_a) = \sigma_a \otimes \sigma_a + \lambda_q^2 T_a \otimes S_a && \epsilon^\circ(\sigma_a) = 1 \\ &\Delta^\circ(T_a) = T_a \otimes \tau_a + \sigma_a \otimes T_a && \epsilon^\circ(T_a) = 0 \\ &\Delta^\circ(S_a) = S_a \otimes \sigma_a + \tau_a \otimes S_a && \epsilon^\circ(S_a) = 0. \end{aligned} \tag{63}$$

To make contact with the q-deformed Lorentz algebra of [3] let us define

$$\begin{aligned} &\tau^1 = \left(\tau^3\right)^{-1/4}\tau_a && \sigma^2 = \left(\tau^3\right)^{1/4}\sigma_a \\ &T^2 = \left(\tau^3\right)^{-1/4}T_a && S^1 = \left(\tau^3\right)^{1/4}S_a. \end{aligned} \tag{64}$$

Using the comultiplication rules we find the actions of the non-compact generators on the basis elements $w_{i_1,j_1}^{(l_1)}\bar{w}_{i_2,j_2}^{(l_2)}$:

$$\begin{aligned} &\tau^1\left(w_{i_1,j_1}^{(l_1)}\bar{w}_{i_2,j_2}^{(l_2)}\right) = q^{2i_1}\delta_{i_1,j_1}\delta_{i_2,j_2} \\ &\sigma^2\left(w_{i_1,j_1}^{(l_1)}\bar{w}_{i_2,j_2}^{(l_2)}\right) = q^{-2i_1}\delta_{i_1,j_1}\delta_{i_2,j_2} + \lambda_q^2 q^{l_1-i_1}E_q(l_1,i_1)\,q^{l_2+i_2}E_q(l_2,i_2)\,\delta_{j_1,i_1+1}\delta_{j_2,i_2+1} \\ &T^2\left(w_{i_1,j_1}^{(l_1)}\bar{w}_{i_2,j_2}^{(l_2)}\right) = q^{l_1+i_1}E_q(l_1,i_1)\,\delta_{j_1,i_1+1}\delta_{j_2,i_2} \\ &S^1\left(w_{i_1,j_1}^{(l_1)}\bar{w}_{i_2,j_2}^{(l_2)}\right) = q^{l_2+i_2}E_q(l_2,i_2)\,\delta_{j_1,i_1}\delta_{j_2,i_2+1}. \end{aligned} \tag{65}$$

By direct inspection one finds the full algebra involving the non-compact q-deformed Lorentz generators:

$$\begin{aligned} &T^+\tau^1 = \tau^1T^+ + \lambda_q T^2 && T^2T^+ = q^{-2}T^+T^2 \\ &T^-\tau^1 = q^{-2}\tau^1T^- - \lambda_q S^1 && T^2T^- = T^-T^2 + \lambda_q^{-1}\left(\sigma^2 - \tau^1\right) \\ &T^2\tau^1 = q^2\tau^1T^2 && S^1T^+ = q^2T^+S^1 + \lambda_q^{-1}\left(\tau^3\tau^1 - \sigma^2\right) \\ &S^1\tau^1 = \tau^1S^1 && S^1T^- = T^-S^1 \\ & && S^1T^2 = T^2S^1 \\ &T^+\sigma^2 = \sigma^2T^+ - q^2\lambda_q T^2\tau^3 && \\ &T^-\sigma^2 = q^2\sigma^2T^- + q^2\lambda_q S^1 && \sigma^2\tau^1 = \tau^1\sigma^2 + q\lambda_q^3 S^1T^2 \\ &T^2\sigma^2 = q^{-2}\sigma^2T^2 && \tau^1\tau^3 = \tau^3\tau^1 \\ &S^1\sigma^2 = \sigma^2S^1 && \sigma^2\tau^3 = \tau^3\sigma^2 \\ &T^2\tau^3 = q^{-4}\tau^3T^2 && \\ &S^1\tau^3 = q^4\tau^3S^1. && \end{aligned} \tag{66}$$

The calculation on the basis elements leads directly to

$$\sigma^2\tau^1 - q^2\lambda_q^2 S^1T^2 = 1^\circ = \epsilon. \tag{67}$$

The content of (66), (59), together with the natural constraint, forms the six-generator q-deformation of the Lorentz algebra, as proposed in [3]. It is important to note that the quantity Z constructed in that work by (67) turns out to be the counit acting on each element of the quantum group $SL_q(2,\mathbb{C})$ in our approach.

Our analysis shows that the algebra $\mathbb{C}\langle\tau^3, T^+, T^-, \tau^1, \sigma^2, T^2, S^1\rangle \subset \mathcal{L}_q^\circ$ is the universal enveloping algebra $\mathcal{U}_q sl(2,\mathbb{C})$. The antipodes on the non-compact generators are given by

$$\begin{aligned} S^\circ(\tau^1) &= \sigma^2 & S^\circ(\sigma^2) &= \tau^1 \\ S^\circ(T^2) &= -q^{-2}(\tau^3)^{1/2}T^2 & S^\circ(S^1) &= -(\tau^3)^{-1/2}S^1. \end{aligned} \tag{68}$$

In contrast to [3] two inequivalent star structures of the non-compact generators are possible.

$$\begin{aligned} (\tau^1)^* &= (\tau^3)^{-1/2}\sigma^2 & (\sigma^2)^* &= (\tau^3)^{1/2}\tau^1 \\ (T^2)^* &= -(\tau^3)^{-1/2}S^1 & (S^1)^* &= -q^{-2}(\tau^3)^{1/2}T^2 \\ (\tau^1)^* &= (\tau^3)^{-1/2}\sigma^2 & (\sigma^2)^* &= (\tau^3)^{1/2}\tau^1 \\ (T^2)^* &= -q^{-2}(\tau^3)^{-1/2}S^1 & (S^1)^* &= -(\tau^3)^{1/2}T^2. \end{aligned} \tag{69}$$

The actions of the non-compact generators (65) lead to the restrictions to $\mathcal{A}_q$ and $\bar{\mathcal{A}}_q$. This will be useful for the chiral decomposition of the q-Lorentz algebra:

$$\begin{aligned} \tau^1|_{\mathcal{A}_q} &= (\tau^3)^{-1/2}|_{\mathcal{A}_q} & \tau^1|_{\bar{\mathcal{A}}_q} &= 1^\circ|_{\bar{\mathcal{A}}_q} \\ \sigma^2|_{\mathcal{A}_q} &= (\tau^3)^{1/2}|_{\mathcal{A}_q} & \sigma^2|_{\bar{\mathcal{A}}_q} &= 1^\circ|_{\bar{\mathcal{A}}_q} \\ T^2|_{\mathcal{A}_q} &= \left(q(\tau^3)^{-1/2}T^+\right)|_{\mathcal{A}_q} & T^2|_{\bar{\mathcal{A}}_q} &= 0|_{\bar{\mathcal{A}}_q} \\ S^1|_{\mathcal{A}_q} &= 0|_{\mathcal{A}_q} & S^1|_{\bar{\mathcal{A}}_q} &= \left(-q^{-1}T^-\right)|_{\bar{\mathcal{A}}_q}. \end{aligned} \tag{70}$$

Following (30), the mapping $r^* : \mathcal{L}_q^* \longrightarrow (\mathcal{A}_q \otimes \bar{\mathcal{A}}_q)^*$ is a bijective algebra morphism. It allows us to decompose the action of the q-Lorentz generators into the $\mathcal{A}_q$ and $\bar{\mathcal{A}}_q$ parts, respectively:

$$\begin{aligned} r^*((\tau^3)^{1/2}) &= (\tau^3)^{1/2}\otimes(\tau^3)^{1/2} & r^*(\tau^1) &= (\tau^3)^{-1/2}\otimes 1^\circ \\ r^*(T^+) &= T^+\otimes 1^\circ + (\tau^3)^{1/2}\otimes T^+ & r^*(\sigma^2) &= (\tau^3)^{1/2}\otimes 1^\circ - \lambda_q^2\cdot T^+\otimes T^- \\ r^*(T^-) &= T^-\otimes 1^\circ + (\tau^3)^{1/2}\otimes T^- & r^*(T^2) &= q\cdot((\tau^3)^{-1/2}T^+)\otimes 1^\circ \\ & & r^*(S^1) &= -q^{-1}\cdot 1^\circ\otimes T^-. \end{aligned} \tag{71}$$

Because of the properties of r^* we can check the relations (66) in $\mathcal{A}_q^\circ \otimes \bar{\mathcal{A}}_q^\circ$.

We now introduce a chiral decomposition of the q-deformed Lorentz algebra. We therefore define the following operators in $\mathcal{L}_q^\circ$:

$$\begin{aligned} N^+ &:= \tau^1T^+ - q^{-1}T^2 & M^+ &:= q^{-1}(\tau^3)^{1/2}T^2 \\ N^- &:= -qS^1 & M^- &:= (\tau^3)^{1/2}(\tau^1T^- + qS^1) \\ N^3 &:= \lambda_q^{-1}\left(1^\circ - \tau^3(\tau^1)^2\right) & M^3 &:= \tau^3\left((\tau^1)^2T^3 - N^3\right). \end{aligned} \tag{72}$$

The M's and N's have the proper classical limits. Their restrictions to $\mathcal{A}_q \otimes \bar{\mathcal{A}}_q$ can be obtained using (71):

$$\begin{aligned} r^*(M^+) &= T^+\otimes(\tau^3)^{1/2} & r^*(N^+) &= 1^\circ\otimes T^+ \\ r^*(M^-) &= T^-\otimes(\tau^3)^{1/2} & r^*(N^-) &= 1^\circ\otimes T^- \\ r^*(M^3) &= T^3\otimes\tau^3 & r^*(N^3) &= 1^\circ\otimes T^3. \end{aligned} \tag{73}$$

This is a chiral decomposition of the q-deformed Lorentz algebra. We see from (73) that the M's and N's have the same algebraic relations as (59), i.e. they seperately form a deformed algebra belonging to $SU_q(2)$. Since the the M's still have a τ^3 on the right side of the tensor sign the two algebras do q-commute, which means for example that $M^+N^+ = q^2N^+M^+$. However, this has no deep consequences in the representation theory since it still holds that $M^3N^3 = N^3M^3$.

We define operators $C_M, C_N \in \mathcal{L}_q^\circ$ by

$$C_M := \frac{q^{-1}\tau^1 + q\sigma^2 - q - q^{-1}}{\left(q - q^{-1}\right)^2} + T^-T^2 \tag{74}$$

$$C_N := \frac{q\left(\tau^3\right)^{1/2}\tau^1 + q^{-1}\left(\tau^3\right)^{-1/2}\sigma^2 - q - q^{-1}}{\left(q - q^{-1}\right)^2} - \left(\tau^3\right)^{-1/2} S^1T^+. \tag{75}$$

These operators are Casimir elements of $\mathcal{U}_q sl(2, \mathbb{C})$. They admit the chiral decomposition

$$r^*(C_M) = C \otimes 1^\circ \qquad r^*(C_N) = 1^\circ \otimes C \tag{76}$$

where C denotes the Casimir element of $\mathcal{U}_q$ introduced in (52). As it should be, these Casimirs interchange under both possible complex conjugations:

$$C_M^* = C_N \qquad \text{and} \qquad C_N^* = C_M. \tag{77}$$

A similar chiral decomposition has been proposed in [3]. However, the generators of that algebra have the undesirable property that the non-compact generator τ^1 has to be inverted, which is not a well defined operation.

The Hopf structure of the chiral generators (72) can be obtained using their defining relations.

7. Spinor bases for the q-deformed Poincaré algebra

The q-deformed Poincaré algebra of [4, 20] can be obtained by adding an inhomogenous part to the q-deformed Lorentz algebra which consists of the comodule algebra of the vector corepresentation of the QLGr, i.e. $V^{\mathrm{L}}(1/2, 1/2)$. This leads to a q-deformed Minkowski four vector [1] generated by coordinates A, B, C and D. The generating relations of the inhomogeneous part are:

$$\begin{aligned} AB &= BA - q^{-1}\lambda_q CD + q\lambda_q D^2 \qquad & BC &= CB - q^{-1}\lambda_q BD \\ AC &= CA + q\lambda_q AD & BD &= q^2 DB \\ AD &= q^{-2}DA & CD &= DC. \end{aligned} \tag{78}$$

To complete the q-deformed Poincaré algebra the actions of the q-Lorentz generators on the four-vector components have to be specified. These relations can be recovered using the results of the previous section. The length of the q-Minkowski vector, $M^2 = q^{-2}CD - AB$, plays the role of the Casimir element, which corresponds to the mass.

In [20, 21] unitary irreducible massive and massless representations of the q-Poincaré algebra have been constructed. We will consider here only the massive case [20]. In this case the states are classified by the eigenvalue of M^2 and labelled by the real eigenvalues of the energy- and z-component of the q-four vector: $P^0 = q(q + q^{-1})^{-1}(C + D)$ and $P^z = (q + q^{-1})^{-1}(qD - q^{-1}C)$ respectively, the third component l of the orbital angular momentum operator T^3, and an additional parameter r which takes values 0 or 1. A general Hilbert space state $|n, N, l, r, F\rangle =: |\mathcal{P}\rangle$ is labelled by the integer eigenvalues of the diagonal generators:

$$\begin{aligned} m^2 &= d_0^2 q^{2F} & p^0 &= d_0 \frac{q^{1-r}}{q+q^{-1}} \left(q^{2(N+1)} + q^{2(F-N+r)}\right) \\ t_3 &= q^{-1}[2l]_{q^{-2}} & p^z &= d_0 \frac{q^{-r}}{q+q^{-1}} \left(q^{2n} - \frac{q^{2(N+1)} + q^{2(F-N+r)}}{q^2+1}\right). \end{aligned} \tag{79}$$

Orthogonality can be defined by $\langle \mathcal{P}'|\mathcal{P}\rangle = \delta_{\mathcal{P}'\mathcal{P}}$.

The analysis in [20] showed that the stability subgroup which induces the massive representations of the q-deformed Poincaré algebra is $SU_q(2)$. However, it was not possible to assign a spin degree of freedom to the corresponding q-deformed one-particle states.

As in the undeformed case we introduce spinor bases. Mathematically speaking, this means we work with covariant rather than with Wigner or Mackey states. Therefore we tensor an arbitrary finite-dimensional representation of $SL_q(2, \mathbb{C})$ to the spinless state vector $|\mathcal{P}\rangle$. A general state in the spinor representation is then

$$|\mathcal{P}, \xi_{i_1,i_2}^{(l_1,l_2)}\rangle = |\mathcal{P}\rangle \otimes \xi_{i_1}^{(l_1)} \bar{\xi}_{i_2}^{(l_2)}. \tag{80}$$

It is easy to see how fields of dotted and undotted spinors can be recovered from (80). The actions of the Poincaré generators on the spinor bases are obtained using their coproducts. Let T denote an arbitrary generator of the q-Poincaré algebra. Then the action on a spinor field is given by

$$T|\mathcal{P}, \xi_{i_1,i_2}^{(l_1,l_2)}\rangle = \Delta(T)\big(|\mathcal{P}\rangle \otimes \xi_{i_1}^{(l_1)} \xi_{i_2}^{(l_2)}\big). \tag{81}$$

We remark that up to now a Hermitian coproduct of the generators of the inhomogenous part of the algebra has not been found. Nevertheless we can assume that the coproduct of the momenta is of the form $\Delta(P^I) = P^I \otimes 1 + \mathcal{O}^I_J \otimes P^J$ [4]. It is clear that the action of the momenta on a pure $SL_q(2, \mathbb{C})$ corepresentation space gives zero.

Specifying the proper inner product of the spinor bases leads to the construction of q-deformed relativistic wave equations for arbitrary spin. This procedure, together with physical applications, is reported in a separate publication [22].

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